These first two pages are reminders of parts of section 3.5 that I had previously indicated you all ought to read. They are relevant for the stuff below on matrix equations.

Recall that the $n \times n$ **identity matrix** is the diagonal matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{1}$$

having ones on its main diagonal and zeros elsewhere. It is not difficult to deduce directly from the definition of the matrix product that $\mathbf{I}$ acts like an identity for matrix multiplication:

$$\mathbf{AI} = \mathbf{A} \quad \text{and} \quad \mathbf{IB} = \mathbf{B} \tag{2}$$

if the sizes of $\mathbf{A}$ and $\mathbf{B}$ are such that the products $\mathbf{AI}$ and $\mathbf{IB}$ are defined. It is, nevertheless, instructive to derive the identities in (2) formally from the two basic facts about matrix multiplication that we state below. First, recall that the notation

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \cdots & \mathbf{a}_n \end{bmatrix} \tag{3}$$

expresses the $m \times n$ matrix $\mathbf{A}$ in terms of its column vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n$.

Fact 1 **Ax** *in terms of columns of* **A**

If $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is an n-vector, then

$$\mathbf{Ax} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n. \tag{4}$$

The reason is that when each row vector of $\mathbf{A}$ is multiplied by the column vector $\mathbf{x}$, its jth element is multiplied by x_j.

Fact 2 **AB** *in terms of columns of* **B**

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$ is an $n \times p$ matrix, then

$$\mathbf{AB} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_p \end{bmatrix}. \tag{5}$$

That is, *the jth column of* **AB** *is the product of* **A** *and the jth column of* **B**. The reason is that the elements of the jth column of $\mathbf{AB}$ are obtained by multiplying the individual rows of $\mathbf{A}$ by the jth column of $\mathbf{B}$.

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The third column of the product **AB** of the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 7 & 5 & -4 \\ -2 & 6 & 3 & 6 \\ 5 & 1 & -2 & -1 \end{bmatrix}$$

is

$$\mathbf{Ab}_3 = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}.$$

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To prove that $\mathbf{AI} = \mathbf{A}$, note first that

$$\mathbf{I} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}, \tag{6}$$

where the jth column vector of **I** is the jth **basic unit vector**

$$\mathbf{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th entry.} \tag{7}$$

If $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, then Fact 1 yields

$$\mathbf{Ae}_j = 0 \cdot \mathbf{a}_1 + \cdots + 1 \cdot \mathbf{a}_j + \cdots + 0 \cdot \mathbf{a}_n = \mathbf{a}_j. \tag{8}$$

Hence Fact 2 gives

$$\begin{aligned} \mathbf{AI} &= \mathbf{A}\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Ae}_1 & \mathbf{Ae}_2 & \cdots & \mathbf{Ae}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}; \end{aligned}$$

that is, $\mathbf{AI} = \mathbf{A}$. The proof that $\mathbf{IB} = \mathbf{B}$ is similar. (See Problems 41 and 42.)

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Matrix Equations

In certain applications, one needs to solve a system $\mathbf{Ax} = \mathbf{b}$ of n equations in n unknowns several times in succession—with the same $n \times n$ coefficient matrix $\mathbf{A}$ each time, but with different constant vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$ on the right. Thus we want to find solution vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ such that

$$\mathbf{Ax}_1 = \mathbf{b}_1, \quad \mathbf{Ax}_2 = \mathbf{b}_2, \quad \ldots, \quad \mathbf{Ax}_k = \mathbf{b}_k. \tag{17}$$

By Fact 2 at the beginning of this section,

$$\begin{bmatrix} \mathbf{Ax}_1 & \mathbf{Ax}_2 & \cdots & \mathbf{Ax}_k \end{bmatrix} = \mathbf{A}\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_k \end{bmatrix}.$$

So the k equations in (17) are equivalent to the single matrix equation

$$\mathbf{AX} = \mathbf{B}, \tag{18}$$

where

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_k \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix}.$$

If $\mathbf{A}$ is invertible and we know $\mathbf{A}^{-1}$, we can find the $n \times k$ matrix of "unknowns" by multiplying each term in Equation (18) on the left by $\mathbf{A}^{-1}$:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}. \tag{19}$$

Note that this equation is a generalization of Eq. (13) in Theorem 4. If $k = 1$, it usually is simplest to solve the system by Gaussian elimination, but when several different solutions are sought, it may be simpler to find $\mathbf{A}^{-1}$ first and then to apply (19).

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Find a 3×4 matrix $\mathbf{X}$ such that

$$\begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 3 & -1 & 2 & 6 \\ 7 & 4 & 1 & 5 \\ 5 & 2 & 4 & 1 \end{bmatrix}.$$

The coefficient matrix is the matrix $\mathbf{A}$ whose inverse we found in Example 7, so Eq. (19) yields

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 & 6 \\ 7 & 4 & 1 & 5 \\ 5 & 2 & 4 & 1 \end{bmatrix},$$

and hence

$$\mathbf{X} = \begin{bmatrix} -4 & -13 & 14 & 1 \\ -1 & -5 & 8 & -2 \\ 11 & 33 & -39 & 4 \end{bmatrix}.$$

By looking at the third columns of $\mathbf{B}$ and $\mathbf{X}$, for instance, we see that the solution of

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 &= 2 \\ 5x_1 + 6x_2 + 3x_3 &= 1 \\ 3x_1 + 5x_2 + 2x_3 &= 4 \end{aligned}$$

is $x_1 = 14$, $x_2 = 8$, $x_3 = -39$. ■